2017-18 AY Assessment Report
B.A. in Mathematics.

Department and Degree: Department of Mathematics. Bachelor's of Arts in Mathematics
Assessment Coordinator: Oscar Vega.

1. **Learning outcomes assessed this year**

Direct Assessment.

We assessed one of our three goals (Goal B) longitudinally. That is, we looked at the progression of courses from the introduction of the concept of proof (MATH 111), to when it gets reinforced (MATH 165), to when it is mastered (MATH 171), all this with a focus on the general area of Mathematical Analysis.
We assessed MATH 111 and MATH 171 in both semesters of the 2017-18 AY, and MATH 165 in Fall 2017 (this class is only offered in the fall).

Next is the description of Goal B and of its SLOs:

Goal B. Communicating Mathematics. Teach students to read, understand, and write rigorous mathematical proofs.

Student Learning Outcomes.

- **B1.** Students will be familiar with common notations and proof techniques.
- **B2.** Students will demonstrate their understanding by reconstructing rigorous proofs of elementary theorems in various areas of mathematics.
- **B3.** Students will be able to write elementary proofs.

Indirect Assessment.

We surveyed graduating students' perceptions of our program, and department.

Other.

We used Tableau to look at the performance, after taking MATH 111 (where proofs are introduced to our majors, Goal B), of our students in the two courses that our majors have the most difficulty with: MATH 151 and MATH 171.

2. **Assignments and/or surveys used to assess the outcomes and criteria/rubrics used for evaluation.**

   For the direct assessment of Goal B: for each of the SLOs (B1, B2, and B3) we embedded questions in the final exams of three courses as follows:

   - A total of three embedded questions on MATH 111 final exams in the 2017-18 AY.
   - A total of two embedded questions on MATH 171 final exams in the 2017-18 AY.
   - One embedded question on the MATH 165 final exam in Fall of 2017.

   Our current SOAP listed only the assessment of MATH 171. We decided to assess a goal instead of assessing just a course; while doing this process, we assessed MATH 171 anyway.

   We use embedded questions in final exams of these three courses to see students' progress in their capacity to read and ability to write proofs, from when students are introduced to proofs in MATH 111, to when they get reinforced in MATH 165, to when they are supposed to be mastered (in Analysis) by students in MATH 171.
   On all SLOs we expected at least 70% of students to achieve a passing score (which depends on the instructor's standards) on every question assessed.

   For the indirect assessment, exit interviews and online exit surveys of students in the graduating class were used to capture students' feelings and thoughts about our program and department.

   We are currently working on creating a new SOAP (for our B.S. program, which will replace our current B.A. program starting in Fall 2018). An alumni survey will be created as a part of this process which we will implement starting next year.

3. **Discoveries from data gathered**

   <u>MATH 111:</u>
   We assessed four sections in the 2017-18 AY; two of them had the same final, and so we will say that we assessed two sections in the Fall and just one in the Spring.

   Section 1. A total of 33 students took this exam. Twenty-four of them passed the exam (5 As, 13 Bs, and 6 Cs), and twenty-four of them passed the class. The mean for the final was a 76% (a C).
   The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were B1: 65% (a D), B2: 80% (a B), and B3: 75% (a C).
   Moreover, **using the passing standard set by the instructor (70%), we got that the percentage/number of students performing at or above that benchmark on each question was 45% (15 students), 76% (25 students), and 67% (22 students),**

respectively. Hence, the class performed above the expected level only for SLO B2, although it was close to meeting expectations in SLO B3 as well.

Of the three embedded questions the one for SLO B2 was, likely, the most predictable of the three, and the scores reflect this assessment. The question for SLO B3 required students to set up and execute a multi-step plan, and the question for SLO B1 dealt with inequalities and asked students to also carry out a plan/proof that required many steps.

Section 2. A total of 20 students took this exam. Ten of them passed the exam (4 As, 2 Bs, and 4 Cs), and twelve of them passed the class. The mean for the final was 68% (a D).
The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were B1: 80% (a B), B2: 83% (a B), and B3: 55% (an F).
Moreover, **using the passing standard set by the instructor (70%), we got that the percentage/number of students performing at or above that benchmark on each question was 80% (16 students), 85% (17 students), and 55% (11 students), respectively. Hence, the class performed above the expected level for SLOs B1 and B2 and considerably below expected for SLO B3.**

Of the three embedded questions, the one for SLO B1 did not ask students to write a proof but to only set up a proof; the question for SLO B2 did require students to write a proof and it is a type of question students would, most probably, expect to be tested on. The other question required students to apply a definition and set up a procedure to obtain the desired result; the conditions of the problem (e.g. domain of the function) may be the reason students did not perform as well in this question as in the other two.

Section 3. A total of 36 students took this exam. Nineteen of them passed the exam (7 As, 10 Bs, and 2 Cs), and twenty-nine of them passed the class. The mean for the final was 68% (a D).
The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were B1: 90% (an A), B2: 67% (a D), and B3: 73% (a C).
Moreover, **using the passing standard set by the instructor (70%), we got that the percentage/number of students performing at or above that benchmark on each question was 89% (32 students), 58% (21 students), and 69% (25 students), respectively. Hence, the class performed above the expected level only for SLO B1, although it was close to performing as expected in SLO B3 as well.**

Of the three embedded questions, the one for SLO B1 did not ask students to write a proof but to only set up a proof; the question for SLO B2 did require students to write a proof and it is a type of question students would, most probably, expect to be tested on. The other question required students to apply a definition and set up a procedure to obtain the desired result; the conditions of the problem (e.g. domain of the function) were simpler than those in the question for SLO B3 for Section 2; this change yielded better scores than in Section 2.

MATH 165:
We assessed one section of this course in the Fall of 2017.

A total of 10 students (out of 12 in the roster) took this exam. Nine of them passed the exam (4 As, 1 B, and 4 Cs), and seven of them passed the class. The mean for the final was 85% (a B).
The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were B1: 85% (a B), B2: 74% (a C), and B3: 61% (a D).
Moreover, **using the passing standard set by the instructor (70%), we got that the percentage/number of students performing at or above that benchmark on each question was 70% (7 students), 60% (6 students), and 60% (6 students), respectively. This was exactly on the mark for SLO B1, but lower than expected for SLOs B2 and B3.**

Of the three embedded questions, the one that required students to use their proving skills the most was the one used for SLO B3; the other two were more 'standard'. Thus it is not surprising that students performed worse on the B3 question. Moreover, this indicates that students need better preparation in addressing questions that require them to move away from certain 'comfort zones' created by 'standard' questions.

MATH 171:
We assessed two sections in the 2017-18 AY; one in the Fall and one in the Spring.

Section 1. A total of 22 students took this exam. Sixteen of them passed the exam (2 As, 6 Bs, and 8 Cs), and seventeen of them passed the class. The mean for the final was 67% (a C).
The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were B1: 73% (a C), B2: 53% (a D), and B3: 57% (a D).
Moreover, **using the passing standard set by the instructor (64%), we got that the percentage/number of students performing at or above that benchmark on each question was 90% (20 students), 27% (6 students), and 45% (10 students), respectively. Hence, the class performed above the expected level for SLO B1 and considerably below expectations for the other two SLOs.**

Of the three embedded questions, the one for SLO B1 was something students would, most probably, expect to be tested on. The other questions measured whether they knew the statement and proof of a well-known result, and whether they were able to use an important theorem to prove a result that required the sequential connection of ideas and small proofs. Hence, we see here that students have difficulty in dealing with new proofs and/or proofs that require more than a single step/idea.

Section 2. A total of 27 students took this exam. Twenty of them passed the exam (9 As, 6 Bs, and 5 Cs), and twenty-three of them passed the class. The mean for the final was a 78% (a C).
The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were B1: 77% (a C), B2: 66% (a D), and B3: 94% (an A).
Moreover, **using the passing standard set by the instructor (70%), we got that the percentage/number of students performing at or above that benchmark on each question was 67% (18 students), 52% (14 students), and 85% (23 students), respectively. Hence, the class performed above the expected level only for SLO B3, although it was close to performing as expected in SLO B1 as well.**

Of the three embedded questions, the one for SLO B3 was something students have, likely, seen several times during the program, going all the way back to Calculus 1. The question for SLO B2 required the careful application of important definitions and techniques, one at a time. The question for SLO B1 was something students should suspect they would be tested on. However, unlike the other two questions, it required students to perform various steps that address different concepts. Once again we see that students have difficulty in dealing with proofs that require more than a single step/idea.

Summarizing, we identify the following issues/remarkable facts.

- We can see in our comments at the end of each course/section performance analysis, that students have difficulties writing proofs that require more than a single step/idea. This is something that we need to address, as students who are not able to develop this skill will, most probably, have difficulty reaching the mathematical maturity expected from a math graduate.
- Questions in which students had to <u>write</u> proofs using/manipulating inequalities proved to be difficult. The percentage of students getting a C or better on such questions was:
  - MATH 111 Fall Section 1, question B1 (45%).
  - MATH 171 Fall, question B3 (45%).
  - MATH 171 Spring, question B1 (67%).

  We contrast these figures with the percentage of students getting a C or better in:
  - MATH 111 Fall Section 2, question B1 (80%).
  - MATH 111 Spring, question B1 (89%).

where students had to deal with an inequality but did not have to write an argument explaining their thoughts.

The following findings come exclusively from the student exit surveys/interviews.

- The exit interviews/surveys are fairly positive, although there is room for improvement. Students seem to feel that the major requires a lot of hard work but that it is rewarding at the end. The instances where criticism was given are:
  (a) Proofs in MATH 128 and MATH 152: Students felt it was unfair for them to have to write proofs in these classes, as MATH 111 is not a prerequisite. They see as a positive thing to see proofs in class.

  (b) Advising and Support Available: Most students agreed that when they looked for advising and/or support they were able to find it. Now, they mention this as if it were well-kept a secret that they want to share with those who are not 'in the know'. We deduce from this that we should do a better job in getting students comfortable asking for advice/help starting in their first semester in the major.

  (c) Mathematical Software: Students felt that more mathematical software should be used in courses; they realize it is something they will need in the future.

  (d) Variations in Instructors' Standards: Students mentioned that there seems to be a wide spectrum in the standards held by instructors, this can be seen at the time of taking a subsequent class but also, sometimes, even for the same class. This creates some uncertainty among students in what to expect from a class. We do not feel these comments should be used to lower standards but to upheld high standards across the board.

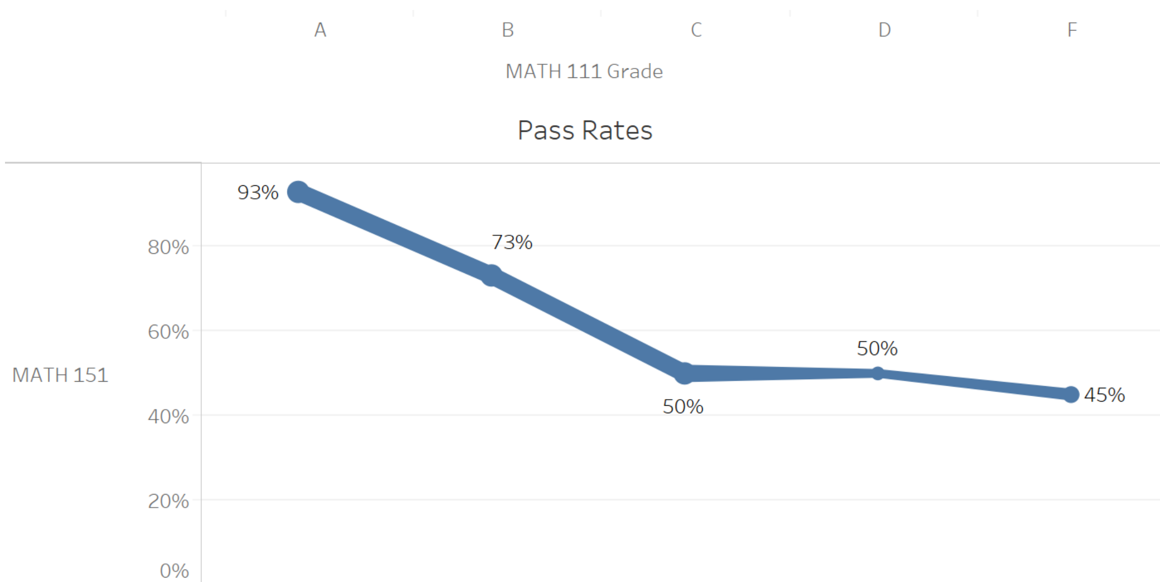
  (e) Career Paths, Graduate School, etc: It was mentioned that we should do a better job in informing students about what they can do with their degree; this pertains not only to careers but also to graduate degrees in the mathematical sciences.

  (f) Road Maps: Students suggested that we create road maps in which we provide more information about what 'desirable prerequisites' every class has.

  (g) Lecturing vs. Other Instruction Styles: Students were split about liking or not liking 'flipped classroom' type of instruction. Most of them liked it but some (coincidentally, all but one of those going to graduate programs) felt that they could have learned the material more effectively on their own. Regarding class activities, it was mentioned that having some kind of quiz at the end of activities would be good to get some 'closure' and to give students the feeling that the time they spent was worthy (this would also help with student motivation).

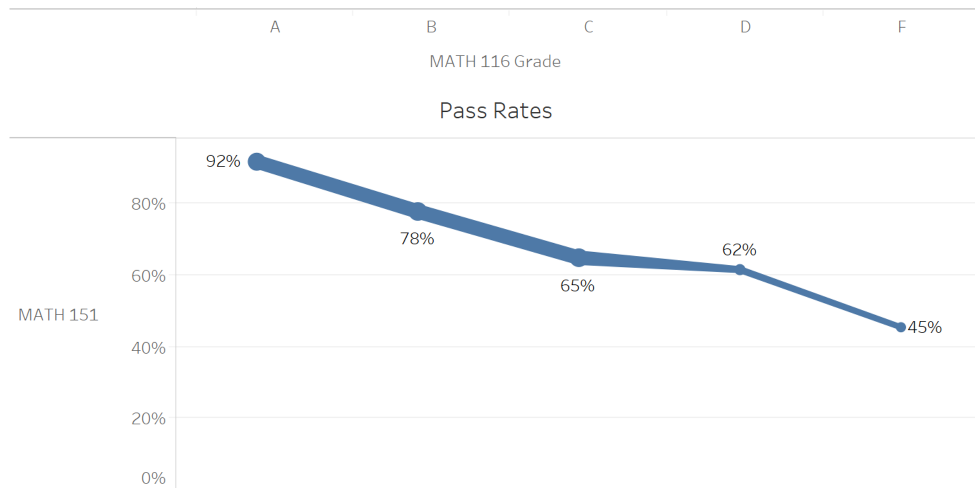
The following findings come exclusively from the Tableau analysis of courses. The data gathered is about students taking MATH 111 for the first time in the range Spring 2011 - Fall 2017, which consists of more than 600 students.

When we look at how MATH 111 'prepares' students for MATH 151, we see that students passing MATH 111 with a C only have a 50% chance of passing MATH 151 (on their first try).

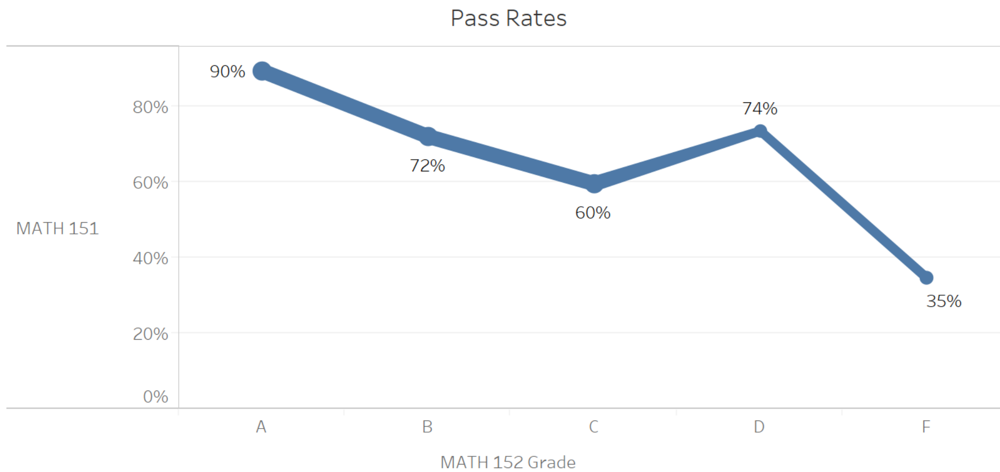


MATH 111 to MATH 151

When we looked at how other courses could complement MATH 111 in the preparation for MATH 151, we looked at MATH 116 and MATH 152. Tableau yielded a much more positive outlook than the one MATH 111 gives by itself.



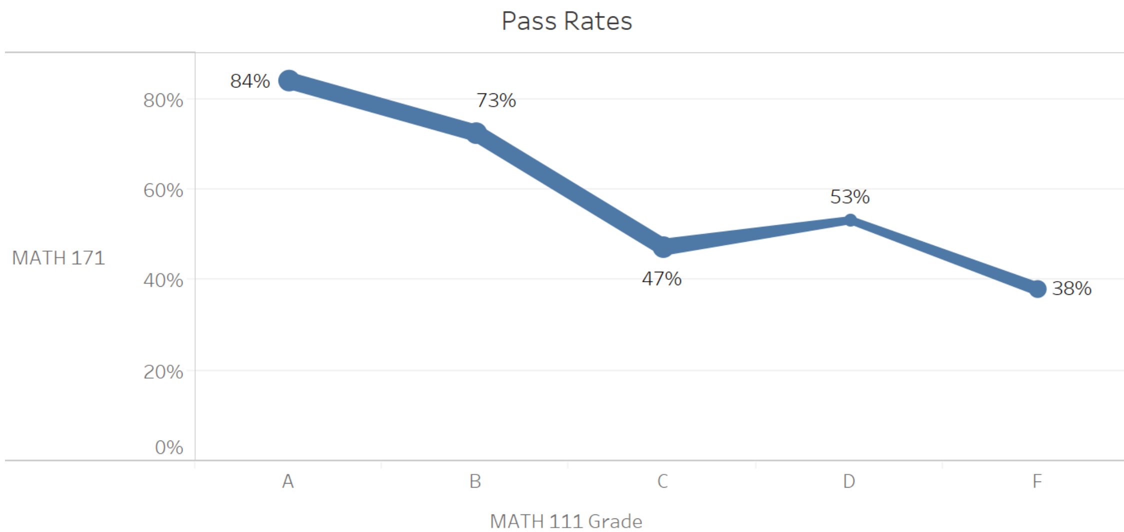
MATH 116 to MATH 151



MATH 152 to MATH 151

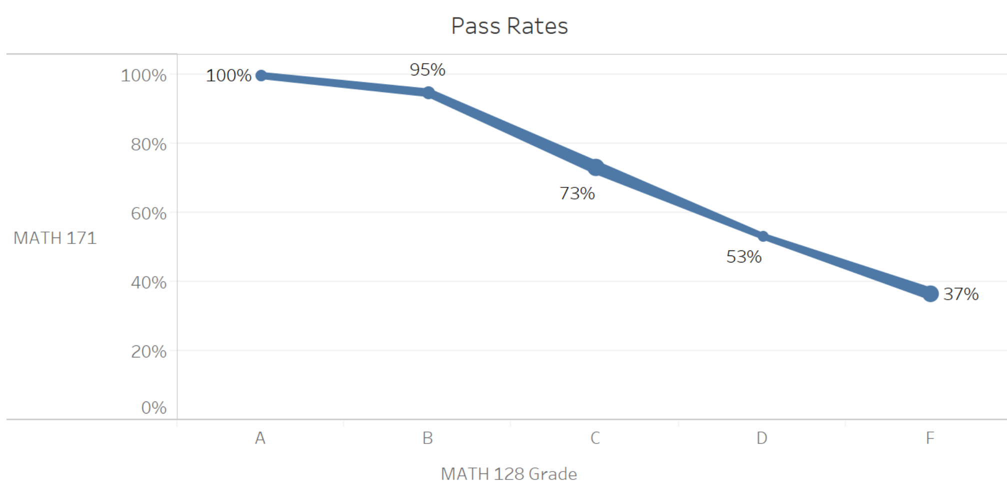
Hence, it makes sense to suggest to our students that they take MATH 116 and MATH 152, if possible, before taking MATH 151.

Similarly, when we looked at how MATH 111 'prepares' students for MATH 171, we see that students passing MATH 111 with a C only have a 47% chance of passing MATH 171 (on their first try). Surprisingly, students who fail MATH 111 with a D (the first time) seem to have a better chance to pass MATH 171 in their first try than those who passed it with a C.
We suspect that this is true because students have more time with learning the language and methods of advanced mathematics.

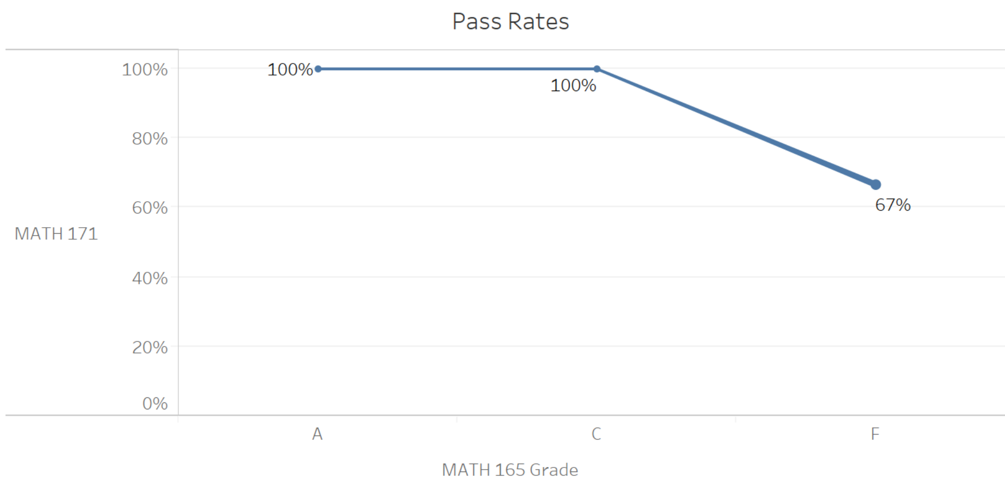


MATH 111 to MATH 171

The two courses that seem to show some better preparation for MATH 171 are MATH 128 and MATH 165.



MATH 128 to MATH 171



MATH 165 to MATH 171

Hence, it makes sense to suggest our students to take MATH 128 and MATH 165, if possible, before taking MATH 171.

4. **Changes made as a result of findings**

While discussing the recommendations in our MA program report, which took place on October, 7th in our inaugural department assessment meeting, we agreed upon the following statement for both graduate and undergraduate courses:

> "Instructors must make sure that they assess all the SLOs their course is supposed to cover. Instructors should find ways to get students to attempt questions that have been written to address specific SLOs."

In that same meeting, and in a department meeting on October, 21st, our department discussed the recommendations below. Our actions and/or conclusions are also listed below.

From the summary at the end of the previous item, we identified four issues that we believe should be addressed by our department:

(i) Advising. There were several comments from students that have relation to advising and guidance about the major, possible careers, and opportunities. These comments were also received in last year's assessment report. We will describe our actions on these issues in Item 6, and regarding to advising specifically, there was a committee formed after last year's recommendation that addressed many of the advising issues mentioned by students (such as designing more informative road maps, and hosting open advising sessions); this committee has been disbanded by now but Dr. Caprau will continue these efforts.

(ii) Proof-writing Training on Analysis Topics. Students have shown difficulties writing proofs that require more than a single step/idea. Also, data shows that students have a hard time working with writing proofs using inequalities (a very important skill to have in a course like MATH 171). Moreover, at least one student mentioned that MATH 111 should have more Analysis in it, and in last year's report we already mentioned that we need to prepare better our students for MATH 171.
We focus our observations on MATH 111 because MATH 171 does not have many courses that would be 'desirable prerequisites' for it. Hence, most of the preparation for this class comes from MATH 111.

Our recommendations are that (1) MATH 111 include more writing of proofs using the real numbers and Calculus ideas; asking for a description of a plan of a proof seems to be insufficient when it comes to training students in constructing/writing proofs; and that (2) classes like MATH 128 and MATH 165 (and, maybe, MATH 161) be revisited so they can 'help' MATH 111 in the preparation of students for MATH 171.

In our departmental discussion, it was clear that our colleagues understood what the issues were and agreed on the need to address them. This means that whoever teaches the classes that deal with, for example, inequalities will make sure that that material is, not only covered, but also assessed and thoroughly discussed. In fact, at least two of the classes assessed last year will feature a stronger preparation for students in the areas discussed above. We did not consider necessary to go beyond this understanding to address this issue.

(iii) Usage of Technology. As it was mentioned in last year's report, we should include more technology in our courses, both in our delivery and in what is requested from our students.
Some work has been done about this (see Item 6), but not enough seemingly, as students keep bringing this up.

In our departmental assessment meeting, it was decided that a committee will be formed to explore the possibility to have a 1-unit course offered to guarantee technology literacy for all our students. Also, individual efforts to include the usage of technology by students is encouraged and it was agreed that it, if present in a class, it should be clearly mentioned in syllabi.

(iv) Clearer Expectations in Courses. We should have clear standards and expectations for the courses we teach. These expectations should be consistent with the course descriptions and requirements. Two instances where we could improve are the requirements of writing proofs in courses where MATH 111 is not required (see Item 6 for some action taken this year regarding MATH 152), and the variability in course standards depending on who is teaching a course in any given semester.
Our department considered impossible to get a unifying view for all courses that we

teach, and considered that trying to control how faculty teach their classes would be overreach and a clearly at odds with academic freedom. Also, we thought that students comments on how we teach our classes, although they make appearances in student evaluations and exit surveys, may be misinformed; the assessment committee will continue monitoring comments of this type students make in the exit surveys.

5. **Assessment activities to be conducted in the 2018-19 academic year.**

All measures (1-9) will be carried out this year, if data is available. Regarding Measure 1, the courses to be assessed will be:

MATH 81, MATH 101, MATH 111, and MATH 151.

Other activities that will also need to be carried out this year follow.

(a) Creation of rubric for senior projects and final presentations (Measure 2).
(b) Creation of rubric for MATH 149S (Measure 3).
(c) Creation of alumni and employer surveys (Measure 4).
(d) Modify current exit survey to assess students' opinions about their culminating experiences (Measure 6).
(e) Creation of form to gather DFW data from instructors (Measure 7).

6. **Progress made on items from our last program review action plan**

During the 2016-17 AY, we had a site visit to review our programs. The visit occurred on Sept. 28th and 29th, 2016. The review panel consisted of Prof. Kim Morin, Theatre Arts, CSU Fresno, Dr. Saeed Attar, Professor of Chemistry, Director of Honors College, CSU Fresno, and Dr. Ivona Grzegorczyk, Professor and Chair, Department of Mathematics, California State University, Channel Islands.

The panel delivered several recommendations that were already discussed in last year's assessment report. Next we report on the items for which we reported no progress last year.

A. Curriculum Improvements and Vision for the Future.

Recommendation 4. Faculty should discuss a long-term vision for the department.
Last year's response.We have had a couple of retreats, in Fall 2015 and Fall 2016, to discuss this. Besides the implementation of a BS in Mathematics, which includes a 'blended' program for students who want to teach at the High School level, no further actions have

been taken. We have discussed the creation of a 4+1 program in the near future.

This year's response. No progress.

B. Supporting Faculty Research and Workload Issues.

Recommendation (Administration and the Department). Identify sources for long term funding so the program can offer release time or summer stipends to faculty engaging in research and grant-writing activities.
Last year's response. No progress.

This year's response. No progress.

C. Departmental Budget.

Recommendation. Identify College and University funds to be included in the departmental funding base for faculty scholarly activities and curriculum coordination.
Last year's response. No progress.

This year's response. No progress.

D. Improving technology use in mathematics courses.

Recommendation. Rethink delivery of the calculus, statistics, and upper division courses to include updated use of technology and current mathematical software.
Last Year's Response. No progress. Any advances on this regard have been made by individuals; our department does not have a plan for this contingency.

This year's response. MATH 111 now includes the usage of LaTeX in its course description. No further progress in other courses or in departmental policy.

E. Supporting Undergraduate and Graduate Student Research.

Recommendation 1. Rethink the ways to involve undergraduate and graduate students into original research rewarding supervising faculty with adequate work load.
Last year's response. No progress on the 'reward' end. Our department's student research committee has created a simple system to identify students who want to engage in research and to match them with faculty willing, and able, to mentor them.

Also, students in our upcoming BS program will have to have a culminating experience, which will involve seminars and, in certain options, a research senior project.

This year's response. No progress on the 'reward' end.

Recommendation 2. Create funding for the department to support small courses for faculty student research projects.
Last year's response. No progress.

This year's response. No progress.

Recommendation 3. Explore the possibility to offer research courses, where full course load is given to faculty for working with small groups of students.
Last year's response. No progress.

This year's response. No progress.

F. Facilities.

Recommendation (Administration) 1. Try to locate all faculty and graduate student offices in closer proximity to the department.
Last year's response. Our Dean has provided three additional offices for our part time faculty to share. Even after this, our need for part time faculty and TA office space remains severe.

This year's response. No progress.

Recommendation (Administration) 2. Provide additional space that is equipped appropriately for best practices in teaching mathematics that will facilitate faculty/student collaboration and research activities.
Last year's response. No progress.

This year's response. No progress.

Recommendation (Administration) 3. Investigate the use of laptops to meet the computing needs of the faculty and students.
Last year's response. All full-time faculty have laptops provided by our college. There has been no progress regarding part-time faculty and students (including our TAs).

This year's response. No progress.

G. Involving Lecturers in Departmental Activities.

Recommendation 1. Include lecturers in programmatic issues relevant to the courses they teach (especially in committees on instruction and curriculum related to their teaching assignments).
Last year's response. No progress.

This year's response. There have been several instances of participation of lecturers in departmental activities.

- A group of lecturers teaching MATH 45 (Jeremy Brandl, Paul Kryder and James Ryan) are working with High School teachers on writing performance tasks for the class. Their work continues this year through an APLU grant, working with graduate students and community college instructors on instructing the MATH 2L sections accompanying MATH 45.
- Przemyslaw Kajetanowicz has participated and gave presentations in seminars, volunteered to conduct GRE prep session, etc.
- Przemyslaw Kajetanowicz and Ganesh Oka worked on the re-design proposal for MATH 75/76.
- Travis Kelm is very active in various departmental events (and on their planning), such as DMD and the Integration Bees.
- Tina Nakashian plays a key role in the coordination of the Single Subject Credential Program, where she also teaches methods classes and supervises student teachers. She also organizes and team-instructs workshops for pre-service and in-service teachers, together with math faculty.
- Lina Obied has helped us in our outreaching efforts with Reedley College, where she also teaches.
- Ravi Somayajulu was a part of the team running the first year of high school level Math Circles.
- Antonina Tofan helped in the coordination of our Teaching Associates.

Other instructors worked on the re-design of graduate-level courses. More details about these activities may be found in our M.A. assessment report.

Recommendation 2. Allocate an additional appropriate space for the program designated to faculty-student collaborations and projects.
Last year's response. No progress.

This year's response. No progress.

H. Assessment and Student Learning Outcomes

Recommendation. The Student Learning Outcomes should include familiarity with current technology accepted by the mathematical community.
Last year's response. No progress. Any advances on this regard have been made by individuals; our department does not have a plan for this contingency.

This year's response. Usage of technology has been added to the SOAP that will start being used when we transition to a B.S. program.

We also report here on the progress made after the assessment committee suggestions in last year's assessment report.

1. Last year we reported that we did not have a plan to 'close the loop'. Since then, our department decided to hold a department meeting early in the Fall semester exclusively dedicated to the discussion of the annual assessment report and to making decisions about actions that would be taken to deal with the suggestions in the assessment report.
   We also said that we will re-write our SOAP, so it fits our new program (In Fall 2018 we will go from offering a B.A. to a B.S.). The SOAP will be discussed and voted upon in the assessment meeting that will take place in early Fall 2018.

2. In last year's report, we suggested MATH 152 to be split into two courses: one of them would be proof-based having MATH 111 as a prerequisite, the other mostly computational. Since then, a committee has been created to design these two courses. We plan to offer MATH 152 (mostly computational) and MATH 153 (proof-based) starting in the Fall of 2019.

3. Students have identified the lack of technology usage requirements in our courses as a big issue. Last year, for the first time ever, workshops on LaTeX were offered at the beginning of the semester, so students got some training on this mathematical typesetting program. Also, MATH 111 now stipulates in its description that LaTeX usage will be required.
   We also said:

   > "We need to expose our students to LaTeX, Mathematica, GeoGebra, Excel, Python, R, etc.
   > Another solution could be to have workshops on technology at the beginning of every semester and/or to have some kind of 'support' that they can access during the semester."

There has been no progress on anything besides LaTeX except for individual efforts by some faculty.

4. In last year's report we suggested that our department take a serious look at advising. Since then, a committee was formed and it has created new road maps, including 'desirable prerequisites'. Our department has also held workshops to help our students switch from our old program (B.A.) to our new one (B.S.).

MATH 111, Assessment Questions and Rubrics. 2017-18 AY

We assessed four sections of this course. The two sections in the Spring were taught by the same instructor, hence we provide questions for Instructors 1, 2, and 3.

The questions, and rubrics, used by the three Instructors follow. We number questions/rubrics according to the label given to each instructor. That is, problems 1 were used by Instructor 1, etc.

SLO B1.

1. Prove that if x, y are nonnegative numbers then $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Rubric. Two examples of full credit answers (12/12 points):
(a) Since x, y are nonnegative,

$$(\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{xy} \geq 0.$$

Thus, $\dfrac{x+y}{2} \geq \sqrt{xy}$.

(b)

$$\begin{aligned}
(x-y)^2 &\geq 0 \\
x^2 - 2xy + y^2 &\geq 0 \\
x^2 + 2xy + y^2 &\geq 4xy \\
(x+y)^2 &\geq 4xy \\
x + y &\geq 2\sqrt{xy} \quad \text{since } x, y \text{ are nonnegative} \\
\text{thus, } \frac{x+y}{2} &\geq \sqrt{xy}.
\end{aligned}$$

Students are expected to assume the inequality in their scratch work, get to the basic inequality $(x-y)^2 \geq 0$ and write the proof backwards.

Example of partial credit (8/12 points):

$$\begin{aligned}
\frac{x+y}{2} &\geq \sqrt{xy} \\
x + y &\geq 2\sqrt{xy} \\
x^2 + y^2 + 2xy &\geq 4xy \\
x^2 - 2xy + y^2 &\geq 0 \\
(x-y)^2 &\geq 0.
\end{aligned}$$

Students lose about 30% of credit if they don't write the proof in the correct order.

2. Consider the following **true** statement:

Let x, $y \in \mathbb{R}$. If x and y are positive numbers, then $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$.

Fill in the appropriate strategies for each type of proof. You do not need to prove the statement.

(a) In order to prove the statement using a direct proof, we would

- suppose that ________________________________

and

- show that ________________________________.

(b) In order to prove the statement using a proof by contrapositive, we would

- suppose that ________________________________

and

- show that ________________________________.

(c) In order to prove the statement using proof by contradiction, we would

- suppose that ________________________________

and

- show that ________________________________.

Rubric. 12 points total, 2 for each line to be filled in.

3. Consider the following **true** statement:

Let $x \in \mathbb{R}$. If $3x^4 + 1 \leq x^7 + x^3$, then $x > 0$.

Fill in the appropriate strategies for each type of proof. You do not need to prove the statement.

(a) In order to prove the statement using a direct proof, we would

- suppose that ________________________________

and

- show that ________________________________.

(b) In order to prove the statement using a proof by contrapositive, we would

- suppose that ________________________________ and

- show that ______________________________.

(c) In order to prove the statement using proof by contradiction, we would

- suppose that ______________________________

and

- show that ______________________________.

Rubric. 18 points total, 3 for each line to be filled in.

SLO B2.

1. Define a relation R on $\mathbb{Z}$ as xRy if and only $4|(x+3y)$.

   (a) Prove that R is an equivalence relation.

   (b) Describe the classes $[1]$ and $[-1]$.

   Rubric.

   (a) Prove that R is an equivalence relation.

   1. Reflexivity [2 points]
   2. Symmetry [3 points]
   3. Transitivity [3 points]

   (b) Describe the classes $[1]$ and $[-1]$.

   1. Description of $[1]$ in a set builder notation [2 points]
   2. Description of $[-1]$ in a set builder notation [2 points]

2. Let A, B, and C be sets. Prove that if $A \subseteq (B \cup C)$ and $A \cap B = \emptyset$, then $A \subseteq C$.

   Rubric. 12 points:

Proof Component	**Points**
Let A, B, C be sets and suppose $A \subseteq (B \cup C)$ and $A \cap B = \emptyset$.	3*
Let $x \in A$.	2
Then $x \in B \cup C$ since $A \subseteq B \cup C$.	2
Thus $x \in B$ or $x \in C$.	1
But $A \cap B = \emptyset$, so x cannot be in both A and B.	2
Thus $x \in C$.	1
Thus $A \subseteq C$.	1*
Total	12

* This total includes points for correct problem setup, format, and strategy. Points are deducted for working backwards or other format issues.

3. Let A, B, and C be sets. Prove that if $C \subseteq A \cup B$, then $C - B \subseteq A$.

Rubric. 15 points:

Proof Component	**Points**
Suppose $C \subseteq (A \cup B)$.	2*
Let $x \in C - B$.	3*
Then $x \in C$ and $x \notin B$.	2
Since $x \in C$ and $C \subseteq A \cup B$ we have $x \in A \cup B$.	3
Thus $x \in A$ or $x \in B$.	2
But $x \notin B$; thus $x \in A$.	3
Thus $C - B \subseteq A$.	
Total	15

* This total includes points for correct problem setup, format, and strategy. Points are deducted for working backwards or other format issues.

SLO B3.

1. The Fibonacci numbers are defined by $f_1 = 1, f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 1$. Now let the Fibonacci-2 numbers g_n be defined as follows: $g_1 = 2, g_2 = 2$, and $g_{n+2} = g_{n+1} \cdot g_n$ for all $n \geq 1$.

   (a) Calculate g_3, g_4, and g_5.

   (b) Show that $g_n = 2^{f_n}$ for all $n \in \mathbb{N}$.

   Rubric.

   (a) 1 point for each correct calculation for a total of 3 points.

   (b) Strong induction:
   Correct base case check and correct induction hypotheses (3 points). Students need to assume $g_{k-1} = 2^{f_{k-1}}$ and $g_k = 2^{f_k}$.
   Complete inductive step (6 points).

2. Let $A = \{2x + 1 \mid x \in \mathbb{Z}\}$ be the set consisting of all the odd integers, and let $g \colon A \to \mathbb{Z}$ be defined by

$$g(n) = \frac{n+1}{2} + 3.$$

You may assume without proof that g is a well-defined function (it is). Prove that g is surjective (onto).

Rubric. 12 points:

Proof Component	**Points**
Let $y \in \mathbb{Z}$. Show that there exists $x \in A$ such that $g(x) = y$.	3*
Consider $x = 2y - 7$.	3**
Note that $x = 2(y-4)+1$, so x is odd. Thus $x \in A$.	2
We have $g(x) = g(2y-7) = \frac{2y-7+1}{2} + 3$	
$= \frac{2y-6+6}{2} = y.$	3
Thus g is onto.	1*
Total	12

* This total includes points for correct problem setup, format, and strategy.
** Points are deducted here if student assumes the existence of x before proving it, etc.

3. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (-1)^n\sqrt{3n+1}$. You may assume without proof that f is a well-defined function (it is). Prove that f is injective (one-to-one).

Rubric. 15 points:

Proof Component	**Points**
Let a, $b \in \mathbb{N}$ and suppose $f(a) = f(b)$.	3*
Then $(-1)^a\sqrt{3a+1} = (-1)^b\sqrt{3b+1}$.	2
(Various ways to show a and b must be the same parity.)	5
Thus $\sqrt{3a+1} = \sqrt{3b+1}$.	2
Thus $3a+1 = 3b+1$; hence $a = b$.	3
Thus f is injective.	
Total	15

* This total includes points for correct problem setup, format, and strategy.

MATH 165, Assessment Questions and Rubrics. 2017-18 AY

We assessed one section of this course in Fall 2017.

The questions, and rubrics, used by the Instructor follow.

SLO B1. For the patch $\mathbf{r}(x, y) = (\cos x, \sin x, y)$, prove that the curve $\boldsymbol{\alpha}(t) = \mathbf{r}(at, bt)$ is a geodesic for any pair of real numbers a and b.

Rubric. We do not know whether $\boldsymbol{\alpha}$ is unit speed, so we better check.

$$\boldsymbol{\alpha}(t) = \mathbf{r}(at, bt) = (\cos{(at)}, \sin{(at)}, bt)$$

then

$$\dot{\boldsymbol{\alpha}}(t) = \frac{d}{dt}(\cos{(at)}, \sin{(at)}, bt) = (-a\sin{(at)}, a\cos{(at)}, b)$$

then

$$\|\dot{\boldsymbol{\alpha}}(t)\| = \|(-a\sin{(at)}, a\cos{(at)}, b)\| = \sqrt{a^2 + b^2}$$

which is not necessarily equal to one, so we cannot assume that $\boldsymbol{\alpha}$ is unit speed. Hence, in order for $\boldsymbol{\alpha}$ to be a geodesic we either reparametrize it by arc length (which could be done and it would yield another, longer, solution) or we prove that $[\dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}}, \mathbf{n}] = 0$.
All that is 4 points

Since we already know $\dot{\boldsymbol{\alpha}}$, we next compute $\ddot{\boldsymbol{\alpha}}$.

$$\ddot{\boldsymbol{\alpha}}(t) = \frac{d}{dt}(-a\sin{(at)}, a\cos{(at)}, b) = (-a^2\cos{(at)}, -a^2\sin{(at)}, 0)$$

Next we find $\mathbf{n}$. We easily compute

$$\mathbf{r}_1(x, y) = (-\sin x, \cos x, 0) \qquad \mathbf{r}_2(x, y) = (0, 0, 1)$$

and thus

$$(\mathbf{r}_1 \times \mathbf{r}_2)(x, y) = (-\sin x, \cos x, 0) \times (0, 0, 1) = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = (\cos x, \sin x, 0)$$

which has length 1. Hence, $\mathbf{n}(x, y) = (\cos x, \sin x, 0)$.
All that is 7 points

At a point on the curve, we get

$$\mathbf{n}(at, bt) = (\cos{(at)}, \sin{(bt)}, 0)$$

All that is 1 point

It is easy to see that $\ddot{\boldsymbol{\alpha}} = -a^2\mathbf{n}$ at every point of $\boldsymbol{\alpha}$ (if students do not see this then they would compute the mix product, which is a long process). Hence, $[\dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}}, \mathbf{n}] = 0$, and thus $\boldsymbol{\alpha}$ is a geodesic. **All that is 3 points**

SLO B2. Let $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\mathbf{r}(x, y) = (x + y, x - y, xy)$.
(a) Show that $\mathbf{r}$ is a simple surface.
(b) Check that all three of Gauβ's formulas hold at $\mathbf{P} = (1, 1, 0)$.

Rubric. (a) We need to prove that $\mathbf{r}$ is injective, C^1 (by time-saving assumption in this exam), and that $\mathbf{r}_1 \times \mathbf{r}_2 \neq \mathbf{0}$.
Assuming $\mathbf{r}(x, y) = \mathbf{r}(u, v)$ yields $(x + y, x - y, xy) = (u + v, u - v, uv)$. Adding the first two components on either side we get $2x = 2u$ and thus $x = u$. Subtracting the first two components yields $y = v$. Hence, $\mathbf{r}$ is injective.
All that is 2 points

$\mathbf{r}$ consists of three component functions, all of them polynomial. It follows that $\mathbf{r}$ is C^∞.
All that is 1 point

We compute the partial derivatives and get

$$\mathbf{r}_1(x, y) = (1, 1, y) \qquad \mathbf{r}_2(x, y) = (1, -1, x)$$

These two vectors are linearly independent, for all $x, y \in \mathbb{R}$, as if $\mathbf{r}_1 = C\mathbf{r}_2$, for some constant C, then $C = 1$ after looking at the first components, but $C = -1$ after looking at the second components. It follows that $\mathbf{r}_1 \times \mathbf{r}_2 \neq \mathbf{0}$.
All that is 2 points

(b) We need to check that

$$\mathbf{r}_{ij} = L_{ij}\mathbf{n} + \Gamma^1_{ij}\mathbf{r}_1 + \Gamma^2_{ij}\mathbf{r}_2$$

for all $i, j = 1, 2$.
We notice that $\mathbf{P} = (1, 0, 1) = \mathbf{r}(1, 0)$, and so

$$\mathbf{r}_1(1, 0) = (1, 1, 0) \qquad \mathbf{r}_2(1, 0) = (1, -1, 1)$$

and so, the $g_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j \rangle$ at $\mathbf{P}$ are

$$g_{11} = 2 \qquad g_{12} = 0 \qquad g_{22} = 3$$

and thus $g = det(g_{ij}) = 6$. Next, we get

$$g^{11} = \frac{g_{22}}{g} = \frac{1}{2} \qquad g^{22} = \frac{g_{11}}{g} = \frac{1}{3} \qquad g^{12} = g^{21} = -\frac{g_{12}}{g} = 0$$

All that is 3 points

Then we compute the second partial derivatives of $\mathbf{r}$ and get

$$\mathbf{r}_{11} = \mathbf{r}_{22} = (0,0,0) \qquad \mathbf{r}_{12} = (0,0,1)$$

We need $\mathbf{n}$.

$$(\mathbf{r}_1 \times \mathbf{r}_2)(1,0) = (1,1,0) \times (1,-1,1) = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = (1,-1,-2)$$

Hence, $\mathbf{n} = \dfrac{(1,-1,-2)}{\sqrt{6}}$.
We next get the $L_{ij} = \langle \mathbf{r}_{ij}, \mathbf{n} \rangle$ at $\mathbf{P}$:

$$L_{11} = L_{22} = (0,0,0) \qquad L_{12} = \frac{-2}{\sqrt{6}}$$

All that is 3 points

Finally, we get the Christoffel symbols at $\mathbf{P}$.

$$\begin{aligned}
\Gamma^k_{11} &= \langle \mathbf{r}_{11}, \mathbf{r}_1 \rangle g^{1k} + \langle \mathbf{r}_{11}, \mathbf{r}_2 \rangle g^{2k} = 0 \cdot g^{1k} + 0 \cdot g^{2k} = 0 \\
\Gamma^k_{22} &= \langle \mathbf{r}_{22}, \mathbf{r}_1 \rangle g^{1k} + \langle \mathbf{r}_{22}, \mathbf{r}_2 \rangle g^{2k} = 0 \cdot g^{1k} + 0 \cdot g^{2k} = 0 \\
\Gamma^1_{12} &= \langle \mathbf{r}_{12}, \mathbf{r}_1 \rangle g^{11} + \langle \mathbf{r}_{12}, \mathbf{r}_2 \rangle g^{21} = 0 \cdot g^{11} + g^{21} = 0 + 0 = 0 \\
\Gamma^2_{12} &= \langle \mathbf{r}_{12}, \mathbf{r}_1 \rangle g^{12} + \langle \mathbf{r}_{12}, \mathbf{r}_2 \rangle g^{22} = 0 \cdot g^{12} + g^{22} = 0 + \frac{1}{3} = \frac{1}{3}
\end{aligned}$$

All that is 3 points

We now check Gauβ's formulas at $\mathbf{P}$.

$$\begin{aligned}
\mathbf{r}_{11} = 0 = 0 \cdot \mathbf{n} + 0 \cdot \mathbf{r}_1 + 0 \cdot \mathbf{r}_2 = L_{11}\mathbf{n} + \Gamma^1_{11}\mathbf{r}_1 + \Gamma^2_{11}\mathbf{r}_2 \\
\mathbf{r}_{22} = 0 = 0 \cdot \mathbf{n} + 0 \cdot \mathbf{r}_1 + 0 \cdot \mathbf{r}_2 = L_{22}\mathbf{n} + \Gamma^1_{22}\mathbf{r}_1 + \Gamma^2_{22}\mathbf{r}_2
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{r}_{12} &= (0,0,1) \\
L_{12}\mathbf{n} + \Gamma^1_{12}\mathbf{r}_1 + \Gamma^2_{12}\mathbf{r}_2 &= \frac{-2}{\sqrt{6}} \frac{(1,-1,-2)}{\sqrt{6}} + 0 \cdot (1,1,0) + \frac{1}{3}(1,-1,1) \\
&= -\frac{(1,-1,-2)}{3} + \frac{(1,-1,1)}{3} \\
&= (0,0,1)
\end{aligned}$$

All that is 1 point

SLO B3. Let $\boldsymbol{\alpha}$ be a regular curve on the image of a coordinate patch $\mathbf{r} : \mathcal{U} \to \mathbb{R}^3$. That is, $\boldsymbol{\alpha}(t) = \mathbf{r}(\alpha_1(t), \alpha_2(t))$. Prove that

$$\|\dot{\boldsymbol{\alpha}}\| = \sqrt{\sum_{i,j=1}^{2} g_{ij}\alpha_i'\alpha_j'}$$

Rubric. Let $\mathbf{r}(\mathcal{U}) = M$. Note that $\boldsymbol{\alpha}$ is a curve of M (not necessarily unit speed). We will compute

$$\|\dot{\boldsymbol{\alpha}}\|^2 = \langle \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle$$

using that $\dot{\boldsymbol{\alpha}} \in T_{\mathbf{P}}M$, and thus using the first fundamental form of M. We know that

$$\langle \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle = I(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}) = \begin{bmatrix} X_1 & X_2 \end{bmatrix} (g_{ij}) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$ are the coordinates of $\dot{\boldsymbol{\alpha}}$ in the basis $\{\mathbf{r}_1, \mathbf{r}_2\}$ of $T_{\mathbf{P}}M$.
All that is 7 points

Since

$$\dot{\boldsymbol{\alpha}} = \frac{d}{dt}\boldsymbol{\alpha} = \frac{d}{dt}\mathbf{r}(\alpha_1(t), \alpha_2(t)) = \mathbf{r}_1\alpha_1' + \mathbf{r}_2\alpha_2'$$

we get that $X_1 = \alpha_1'$ and $X_2 = \alpha_2'$.
All that is 3 points

Hence,

$$\|\dot{\boldsymbol{\alpha}}\|^2 = \langle \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle = I(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}) = \begin{bmatrix} \alpha_1' & \alpha_2' \end{bmatrix} (g_{ij}) \begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix} = \sum_{i,j=1}^{2} g_{ij}\alpha_i'\alpha_j'$$

We obtain the result by taking square roots both sides.
All that is 5 points

MATH 171, Assessment Questions and Rubrics. 2017-18 AY

We assessed two section of this course: one in Fall 2017 and another in Spring 2018.

The questions, and rubrics, used by the two Instructors follow. We number questions/rubrics according to the label given to each instructor. That is, problems 1 were used by Instructor 1, etc.

SLO B1.

1. Prove that $f(x) = \frac{1}{x}$ is differentiable at any point a in its domain.

   Solution The domain of f is all real numbers not equal to zero, so $a \neq 0$. Definition: The derivative of $f(x)$ at $x = a$ is $f'(a)$ when for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \epsilon$.
   Proof: Let $\epsilon > 0$. Choose $\delta = \min(\epsilon * (|a| + 1)^3, 1)$. Suppose $0 < |x - a| < \delta$. Then $|x| \leq |a| + 1$ We have that

$$\begin{aligned}\left|\frac{f(x) - f(a)}{x - a} - \left(\frac{-1}{a^2}\right)\right| &= \left|\frac{\frac{1}{x} - \frac{1}{a}}{x - a} + \frac{1}{a^2}\right| \\ &= \left|\frac{-1}{xa} + \frac{1}{a^2}\right| \\ &= \frac{|x - a|}{|x||a|^2} \\ &< \frac{\delta}{(|a| + 1)^3} \\ &\leq \epsilon\end{aligned}$$

   Rubric: Total points 30.

   - 10 points for correct statement of definition; partial credit for order errors. Points will be given in case where student verifies definition in proof without stating it.
   - 5 points for identifying the derivative value.
   - 10 points for essential arguments of proof (bounds).
   - 5 points for a feasible choice of δ.

2. Use the definition (no other properties are allowed) to prove that

$$\lim_{x \to 1} \frac{1}{x} = 1.$$

   Rubric.

- (6 points) Let $\epsilon > 0$. We will find $\delta > 0$ so that if $0 \neq |x-1| < \delta$, then

$$\left|\frac{1}{x} - 1\right| < \epsilon$$

which is equivalent to

$$\frac{|x-1|}{|x|} < \epsilon.$$

- (7 points) We note that if $|x-1| < 1/2$ then

$$\frac{|x-1|}{|x|} < \frac{|x-1|}{\frac{1}{2}} = 2|x-1|.$$

- (7 points) Choose $\delta = \min(1/2, \epsilon/2)$. If $|x-1| < \delta$ then

$$\frac{|x-1|}{|x|} < \frac{|x-1|}{\frac{1}{2}} = 2|x-1| < 2\frac{\epsilon}{2} = \epsilon.$$

SLO B2.

1. State and prove the Ratio test for a general series $\sum_k a_k$.

Solution: The Ratio test states: "For an infinite series $\sum a_n$, consider the ratio $r(n) = |a_{n+1}/a_n|$. If there exists $\alpha < 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $r(n) \leq \alpha$, then the series $\sum a_n$ converges absolutely."

Proof: Suppose the conditions above. For $n \geq N$, $|a_{n+1}| \leq \alpha |a_n|$. This bound can be iterated back to a_N, so we have $|a_n| \leq \alpha^{n-N}|a_N|$. The partial sums of $n \geq N$ terms is bounded above by:

$$\begin{aligned}\sum_{k=1}^{n} |a_k| &= \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{n} |a_k| \\ &\leq \sum_{k=1}^{N-1} |a_k| + |a_N| \sum_{k=N} \alpha^{n-N}\end{aligned}$$

This bounding series is a convergent geometric series after the first N terms which converges to $\sum_{k=1}^{N-1} |a_k| + \frac{|a_N|}{1-\alpha}$. By the comparison test, the series converges absolutely since the series of absolute is bounded above by a convergent series.

Rubric: Total 30 points.

- 12 points for statement of Ratio test.

- 18 points for proof. This includes using the ratio bound to bound the summands, correct use of the geometric series, and use of the comparison test.

2. (a) Prove that if $f(x)$ is integrable on $[a,b]$ then $f^2(x)$ is integrable on $[a,b]$.
   (b) Prove that if $f(x)$ and $g(x)$ are integrable on $[a,b]$ then $f(x)g(x)$ is integrable on $[a,b]$. Hint: expand $(f+g)^2$ and be clear on which properties you are using.
   (c) Is it possible to have some bounded function $h(x)$ where $h^2(x)$ is integrable on $[a,b]$ but $h(x)$ is NOT integrable on $[a,b]$? If yes, give an example. If no, explain why.
   (d) Is it possible to have a bounded function $h(x)$ where $\sqrt[3]{h(x)}$ is integrable on $[a,b]$ but $h(x)$ is NOT integrable on $[a,b]$? If yes, give an example. If no, explain why.

Rubric.

(a) (5 points) Since $f(x)$ is integrable on $[a,b]$, $|f(x)|$ is integrable $[a,b]$. Because $|f(x)| \geq 0 \ \forall x$, we conclude $|f(x)|^2$ is integrable and thus $f(x)^2 = |f(x)|^2$ is integrable.

(b)
- (3 points) We have

$$(f(x)+g(x))^2 = f(x)^2 + 2f(x)g(x) + g(x)^2$$

and thus

$$f(x)g(x) = \frac{(f(x)+g(x))^2 - f(x)^2 - g(x)^2}{2}.$$

- (2 points) Since the sum of integrable functions is integrable, the square of an integrable function is integrable, and a constant multiple of an integrable function is integrable, we conclude $f(x)g(x)$ is integrable from the equation above.

(c) (5 points, no points are given to a yes/no without explanation).
YES. Consider on $[0,1]$

$$h(x) = f(x) - \frac{1}{2} = \begin{cases} 1/2 & \text{if } x \in \mathbb{Q} \\ -1/2 & \text{else} \end{cases}$$

where $f(x)$ is the Dirichlet function. Since the Dirichlet function is not integrable on $[0,1]$, $h(x)$ is not integrable. However, $h^2(x)$ is the constant $1/4$ function which is integrable.

(d) (5 points, no points are given to a yes/no without explanation)
NO. Suppose $\sqrt[3]{h(x)}$ is integrable. By part (b) the product of two integrable function is integrable. This will imply $h(x) = \sqrt[3]{h(x)}.\sqrt[3]{h(x)}.\sqrt[3]{h(x)}$ is integrable.

SLO B3.

1. State the Mean Value Theorem for derivatives. Use it to prove the following part of the Fundamental Theorem of Calculus: "If $f(x)$ is differentiable on $[a, b]$ and $f'(x)$ is continuous on $[a, b]$, then $\int_a^b f'(x)dx = f(b) - f(a)$. "

   Solution: Mean Value Theorem: "If $f(x)$ is continuous on the interval $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$."

   Proof: Suppose that $f(x)$ is differentiable on $[a, b]$ and $f'(x)$ is continuous on $[a, b]$. It follows that f' is Riemann integrable on $[a, b]$ since it is continuous on $[a, b]$.
   Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ satisfying $x_j - x_{j-1} < \delta$ and any choice of x_j^* in $[x_{j-1}, x_j]$, then

$$|\sum_{j=1}^{n} f'(x*_j)(x_j - x_{j-1}) - \int_a^b f'(x)dx| < \epsilon$$

   Choose such a partition. For each subinterval $[x_{j-1}, x_j]$, f is continuous and differentiable. By the Mean Value Theorem, there exists a x_j^* such that $f'(x_j^*) = \frac{f(x_j)-f(x_{j-1})}{x_j-x_{j-1}}$. From this choice, we obtain

$$\begin{aligned} \sum_{j=1}^{n} f'(x_j^*)(x_j - x_{j-1}) &= \sum_{j=1}^{n} f(x_j) - f(x_{j-1}) \\ &= f(x_n) - f(x_0) \\ &= f(b) - f(a) \end{aligned}$$

   We have for every ϵ, $|(f(b) - f(a)) - \int_a^b f'(x)dx| < \epsilon$. This holds only if $\int_a^b f'(x)dx = f(b) - f(a)$.

   Rubric: Total 30 points.

   - 10 points for correct statement of Mean Value Theorem for derivative.
   - 10 points for correct definition of Riemann integrability.
   - 10 points for correct proof combining the above two in a feasible manner.

2. Assume that the function $\sqrt{x}$ is continuous at any $c > 0$. Prove that the function

$$f(x) = \frac{1}{\sqrt{x}}$$

   is differentiable at any point $c > 0$ and its derivative at c is

$$f'(c) = -\frac{1}{2\sqrt{c^3}}.$$

   Rubric.

- (5 points) Consider any sequence $x_n \to c$ and $x_n \neq c \; \forall n$. We have

$$\frac{f(x_n) - f(c)}{x_n - c} = \frac{\frac{1}{\sqrt{x_n}} - \frac{1}{\sqrt{c}}}{x_n - c}.$$

- (5 points) We simply the right side as

$$\frac{\sqrt{c} - \sqrt{x_n}}{\sqrt{x_n}\sqrt{c}(x_n - c)}$$

- (5 points) We apply square-root conjugate formula and convert this to

$$\frac{c - x_n}{\sqrt{x_n}\sqrt{c}(x_n - c)(\sqrt{c} + \sqrt{x_n})} = -\frac{1}{\sqrt{x_n}\sqrt{c}(\sqrt{c} + \sqrt{x_n})}.$$

- (5 points) Since the square-root function is continuous we have $\sqrt{x_n} \to \sqrt{c}$ and thus the limit of the right side becomes

$$-\frac{1}{2\sqrt{c}^3}.$$